Whitham systems and deformations.

A.Ya. Maltsev

L.D.Landau Institute for Theoretical Physics,
119334 ul. Kosygina 2, Moscow, maltsev@itp.ac.ru

Abstract

We consider the deformations of Whitham systems including the "dispersion terms" and having the form of Dubrovin-Zhang deformations of Frobenius manifolds. The procedure is connected with B.A. Dubrovin problem of deformations of Frobenius manifolds corresponding to the Whitham systems of integrable hierarchies. Under some non-degeneracy requirements we suggest a general scheme of the deformation of the hyperbolic Whitham systems using the initial non-linear system. The general form of the deformed Whitham system coincides with the form of the "low-dispersion" asymptotic expansions used by B.A. Dubrovin and Y. Zhang in the theory of deformations of Frobenius manifolds.

1 Introduction.

The classical Whitham method ([1, 2, 3, 4]) is connected with the slow modulations of the exact periodic or quasiperiodic solutions of non-linear PDE's

$$F^{i}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{t}, \boldsymbol{\varphi}_{x}, \boldsymbol{\varphi}_{tt}, \boldsymbol{\varphi}_{xt}, \boldsymbol{\varphi}_{xx}, \dots) \,=\, 0 \quad , \quad i = 1, \dots, n \tag{1.1}$$

where $\boldsymbol{\varphi} \,=\, (\varphi^{1}, \dots, \varphi^{n})$.

It is assumed that the system (1.1) admits the finite-parametric family of exact solutions

$$\varphi^{i}(x,t) \,=\, \Phi^{i}(\mathbf{k}(\mathbf{U})x \,+\, \boldsymbol{\omega}(\mathbf{U})t \,+\, \boldsymbol{\theta}_{0}, \,\mathbf{U}) \tag{1.2}$$

where $\boldsymbol{\theta} \,=\, (\theta^{1}, \dots, \theta^{m})$, $\Phi^{i}(\boldsymbol{\theta}, \mathbf{U})$ are smooth functions 2π-periodic w.r.t. each θ^{α}, $\mathbf{k}(\mathbf{U}) \,=\, (k^{1}(\mathbf{U}), \dots, k^{m}(\mathbf{U}))$, $\boldsymbol{\omega}(\mathbf{U}) \,=\, (\omega^{1}(\mathbf{U}), \dots, \omega^{m}(\mathbf{U}))$ are "wave numbers" and "frequencies" of the solution, $\mathbf{U} \,=\, (U^{1}, \dots, U^{N})$ are parameters of the solution, and $\boldsymbol{\theta}_{0} \,=\, (\theta_{0}^{1}, \dots, \theta_{0}^{m})$ are arbitrary initial phases.

The functions $\boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ satisfy the nonlinear system

$$F^{i}\left(\boldsymbol{\Phi}, \, \omega^{\alpha}(\mathbf{U}) \, \boldsymbol{\Phi}_{\theta^{\alpha}}, \, k^{\beta}(\mathbf{U}) \, \boldsymbol{\Phi}_{\theta^{\beta}}, \, \omega^{\gamma}(\mathbf{U}) \, \omega^{\delta}(\mathbf{U}) \, \boldsymbol{\Phi}_{\theta^{\gamma}\theta^{\delta}}, \, \dots\right) \quad = \quad 0 \tag{1.3}$$

We can introduce the families $\Lambda_{\mathbf{k},\boldsymbol{\omega}}$ and the full family $\Lambda = \bigcup \Lambda_{\mathbf{k},\boldsymbol{\omega}}$ of the functions $\boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ satisfying the system (1.3) in the space of 2π-periodic w.r.t. each θ^{α} functions. Let us choose (in a smooth way) at every $(U^1, \dots, U^N)$ some function $\boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ as having "zero initial phase shifts" and represent the full family of m-phase solutions of system (1.1) in the form (1.2).

In Whitham method we make a rescaling $X = \epsilon\, x$, $T = \epsilon\, t$ ($\epsilon \rightarrow 0$) of both variables x and t and try to find a function

$$\mathbf{S}(X,T) \;=\; \big(S^1(X,T), \dots, S^m(X,T)\big) \tag{1.4}$$

and 2π-periodic functions

$$\Psi^i(\boldsymbol{\theta}, X, T, \epsilon) \;\;=\;\; \sum_{k\geq 0} \Psi^i_{(k)}(\boldsymbol{\theta}, X, T)\; \epsilon^k \tag{1.5}$$

such that the functions

$$\phi^i(\boldsymbol{\theta}, X, T, \epsilon) \;\;=\;\; \Psi^i\left(\frac{\mathbf{S}(X,T)}{\epsilon} \,+\, \boldsymbol{\theta}, X, T, \epsilon\right) \tag{1.6}$$

satisfy the system

$$F^i\left(\boldsymbol{\phi},\, \epsilon\, \boldsymbol{\phi}_T,\, \epsilon\, \boldsymbol{\phi}_X,\, \epsilon^2\, \boldsymbol{\phi}_{TT},\, \dots\right) \;\;=\;\; 0 \tag{1.7}$$

at every X, T and $\boldsymbol{\theta}$.

It is easy to see that the function $\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T)$ satisfies the system (1.3) at every X and T with

$$k^{\alpha} \;\;=\;\; S^{\alpha}_X \quad , \quad \omega^{\alpha} \;\;=\;\; S^{\alpha}_T$$

and so belongs at every (X,T) to the family Λ. We can write then

$$\Psi^i_{(0)}(\boldsymbol{\theta}, X, T) \;\;=\;\; \Phi^i(\boldsymbol{\theta} \,+\, \boldsymbol{\theta}_0(X,T), \mathbf{U}(X,T))$$

and introduce the functions $U^{\nu}(X,T)$, $\theta^{\alpha}_0(X,T)$ as the parameters characterizing the main term in (1.5) which should satisfy the condition

$$[k^{\alpha}(\mathbf{U})]_T \;\;=\;\; [\omega^{\alpha}(\mathbf{U})]_X \tag{1.8}$$

The functions $\Psi^i_{(1)}(\boldsymbol{\theta}, X, T)$ are defined from the linear system

$$\hat{L}^i_j\, \Psi^j_{(1)}(\boldsymbol{\theta}, X, T) \;\;=\;\; f^i_{(1)}(\boldsymbol{\theta}, X, T) \tag{1.9}$$

where

$$\hat{L}^i_j \;\;=\;\; \hat{L}^i_{(X,T)\,j} \;\;=\;\; \frac{\partial F^i}{\partial \varphi^j}\left(\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T),\, \dots\right) \,+$$

$$+ \frac{\partial F^i}{\partial \varphi^j_t} \left(\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \, \dots \right) \, \omega^{\alpha}(X,T) \frac{\partial}{\partial \theta^{\alpha}} + \frac{\partial F^i}{\partial \varphi^j_x} \left(\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \, \dots \right) \, k^{\beta}(X,T) \frac{\partial}{\partial \theta^{\beta}} + \dots \tag{1.10}$$

is the linearization of system (1.3) and $\mathbf{f}_{(1)}(\boldsymbol{\theta}, X, T)$ is discrepancy given by the expression

$$f^i_{(1)}(\boldsymbol{\theta}, X, T) \; = \; - \frac{\partial F^i}{\partial \varphi^j_t} \left(\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \, \dots \right) \, \Psi^j_{(0)T}(\boldsymbol{\theta}, X, T) \, -$$

$$- \frac{\partial F^i}{\partial \varphi^j_x} \left(\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \, \dots \right) \, \Psi^j_{(0)X}(\boldsymbol{\theta}, X, T) \, -$$

$$- \frac{\partial F^i}{\partial \varphi^j_{tt}} \left(\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T), \, \dots \right) \left(2 \, \omega^{\alpha}(X,T) \, \Psi^j_{(0)\theta^{\alpha} T} \, + \, \omega^{\beta}_T(X,T) \, \Psi^j_{(0)\theta^{\beta}} \right) \, - \, \dots \tag{1.11}$$

We have here

$$\frac{\partial}{\partial T} \; = \; U^{\nu}_T \frac{\partial}{\partial U^{\nu}} \; + \; \theta^{\alpha}_{(0)T} \frac{\partial}{\partial \theta^{\alpha}} \quad , \quad \frac{\partial}{\partial X} \; = \; U^{\nu}_X \frac{\partial}{\partial U^{\nu}} \; + \; \theta^{\alpha}_{(0)X} \frac{\partial}{\partial \theta^{\alpha}}$$

for the functions

$$\Psi^i_{(0)}(\boldsymbol{\theta}, X, T) \;\; = \;\; \Phi^i(\boldsymbol{\theta} \, + \, \boldsymbol{\theta}_0(X,T), \mathbf{U}(X,T))$$

We will assume that k^{α} and ω^{α} can be considered (locally) as the independent parameters on the family Λ and the total family of solutions of (1.3) depends (for generic k^{α}, ω^{α}) on $N \, = \, 2m \, + \, s$, $(s \geq 0)$ parameters U^{ν} and m initial phases θ^{α}_0.

Easy to see that the functions $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta} \, + \, \boldsymbol{\theta}_0(X,T), \mathbf{U}(X,T))$ and $\nabla_{\boldsymbol{\xi}} \, \mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta} \, + \, \boldsymbol{\theta}_0(X,T), \mathbf{U}(X,T))$ where $\boldsymbol{\xi}$ is any vector in the space of parameters U^{ν} tangential to the surface $\mathbf{k} \, = \, const$, $\boldsymbol{\omega} \, = \, const$ belong to the kernel of the operator $\hat{L}^i_{(X,T)\,j}$.

Let us put now some "regularity" conditions on the family (1.2) of quasiperiodic solutions of (1.1)

Definition 1.1.

We call the family (1.2) the full regular family of m-phase solutions of (1.1) if:

1) The functions $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta}, \mathbf{U})$, $\mathbf{\Phi}_{U^{\nu}}(\boldsymbol{\theta}, \mathbf{U})$ *are linearly independent (almost everywhere) on the set* Λ*;*

2) The $m \, + \, s$ *linearly independent functions* $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta}, \mathbf{U})$, $\nabla_{\boldsymbol{\xi}} \, \mathbf{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ $(\nabla_{\boldsymbol{\xi}} \mathbf{k} \, = \, 0$, $\nabla_{\boldsymbol{\xi}} \boldsymbol{\omega} \, = \, 0)$ *give the full kernel of the operator* $\hat{L}^i_{[\mathbf{U}]\,j}$ *(here* $\boldsymbol{\theta}_0 \, = \, 0$*) for generic* $\mathbf{k}$ *and* $\boldsymbol{\omega}$*.*

3) There are exactly $m \, + \, s$ *linearly independent "left eigen-vectors"* $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta})$, $q \, = \, 1, \dots, m + s$ *of the operator* $\hat{L}^i_{[\mathbf{U}]\,j}$ *(for generic* $\mathbf{k}$ *and* $\boldsymbol{\omega}$*) corresponding to zero eigen values i.e.*

$$\int_0^{2\pi} \!\! \dots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{U}]i}(\boldsymbol{\theta}) \, \hat{L}^i_{[\mathbf{U}]\,j} \, \psi^j(\boldsymbol{\theta}) \, \frac{d^m \theta}{(2\pi)^m} \;\; \equiv \;\; 0$$

for any periodic $\psi^j(\boldsymbol{\theta})$.

It's not difficult to see that the Definition 1.1 is connected with the regularity properties of the sub-manifold Λ given by the set of functions $\mathbf{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ in the space of 2π-periodic functions. In fact, for our purposes we can use also the weaker definition of the full regular family of m-phase solutions of (1.1). Namely, let us represent the space of parameters $\mathbf{U}$ in the form $\mathbf{U} = (\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ where $\mathbf{k}$ are the wave numbers, $\boldsymbol{\omega}$ are the frequencies of m-phase solutions, and $\mathbf{n} = (n^1, \dots, n^s)$ are some additional parameters (if they exist). Let us give now the "weak" definition of the full regular family of m-phase solutions in the form:

Definition 1.1′.

We call the family Λ the full regular family of m-phase solutions of (1.1) if

1) The functions $\mathbf{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$, $\mathbf{\Phi}_{n^l}(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ are linearly independent and give (for generic $\mathbf{k}$ and $\boldsymbol{\omega}$) the full basis in the kernel of the operator $\hat{L}^i_{j[\boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}]}$;

2) The operator $\hat{L}^i_{j[\boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}]}$ has (for generic $\mathbf{k}$ and $\boldsymbol{\omega}$) exactly $m+s$ linearly independent "left eigen vectors"

$$\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) \;\; = \;\; \boldsymbol{\kappa}^{(q)}_{[\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0)$$

depending on the parameters $\mathbf{U}$ in a smooth way and corresponding to zero eigen-values.

We will assume now that the system (1.1) has a full regular family of quasiperiodic m-phase solutions in strong or weak sense in all our considerations.

To find the function $\mathbf{\Psi}_{(1)}(\boldsymbol{\theta}, X, T)$ we have to put now the $m + s$ conditions of orthogonality of the discrepancy $\mathbf{f}_{(1)}(\boldsymbol{\theta}, X, T)$ to the functions $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}(X,T)]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X, T))$

$$\int_0^{2\pi} \!\! \dots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{U}(X,T)]\,i}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X, T)) \, f^i_{(1)}(\boldsymbol{\theta}, X, T) \, \frac{d^m \theta}{(2\pi)^m} \;\; = \;\; 0 \qquad (1.12)$$

The system (1.12) together with (1.8) gives $m + (m + s) = 2m + s = N$ conditions at each X and T on the parameters of zero approximation $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T)$ necessary for the construction of the first ϵ-term in the solution (1.5).

Let us prove now the following Lemma about the orthogonality conditions (1.12):

Lemma 1.1.

Under all the assumptions of regularity formulated above the orthogonality conditions (1.12) do not contain the functions $\theta^\alpha_0(X, T)$ and give just the restrictions on the functions $U^\nu(X, T)$ having the form

$$C^{(q)}_\nu(\mathbf{U}) \, U^\nu_T \; - \; D^{(q)}_\nu(\mathbf{U}) \, U^\nu_X \; = \; 0 \qquad (1.13)$$

(with some functions $C^{(q)}_\nu(\mathbf{U})$, $D^{(q)}_\nu(\mathbf{U})$).

Proof.

Let us write down the part $\mathbf{f}'_{(1)}$ of the function $\mathbf{f}_{(1)}$ which contains the derivatives $\theta^{\beta}_{0T}(X,T)$ and $\theta^{\beta}_{0X}(X,T)$. We have from (1.11)

$$f'^{i}_{(1)}(\boldsymbol{\theta},X,T) \; = \; - \frac{\partial F^{i}}{\partial \varphi^{j}_{t}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, \Psi^{j}_{(0)\theta^{\beta}} \, \theta^{\beta}_{0T} \; - \; \frac{\partial F^{i}}{\partial \varphi^{j}_{x}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, \Psi^{j}_{(0)\theta^{\beta}} \, \theta^{\beta}_{0X} \; -$$

$$- \, \frac{\partial F^{i}}{\partial \varphi^{j}_{tt}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, 2\,\omega^{\alpha}(X,T)\, \Psi^{j}_{(0)\theta^{\alpha}\theta^{\beta}} \, \theta^{\beta}_{0T} \; - \; \frac{\partial F^{i}}{\partial \varphi^{j}_{xx}} \left(\boldsymbol{\Psi}_{(0)}, \, \dots \right) \, 2\,k^{\alpha}(X,T)\, \Psi^{j}_{(0)\theta^{\alpha}\theta^{\beta}} \, \theta^{\beta}_{0X} \; - \; \dots$$

Let us choose again the set of parameters $\mathbf{U}$ in the form

$$\mathbf{U} = (k^{1}, \dots, k^{m}, \omega^{1}, \dots, \omega^{m}, n^{1}, \dots, n^{s})$$

where k^{α} are the wave numbers, ω^{α} are the frequencies of m-phase solutions and $(n^{1}, \dots, n^{s})$ are additional parameters (except the initial phases).

We can write then

$$f'^{i}_{(1)}(\boldsymbol{\theta},X,T) \; = \; \left[- \frac{\partial}{\partial \omega^{\beta}} \, F^{i} \left(\boldsymbol{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}), \dots \right) \; + \; \hat{L}^{i}_{j} \, \frac{\partial}{\partial \omega^{\beta}} \, \Phi^{j}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}) \right] \, \theta^{\beta}_{0T} \; +$$

$$+ \; \left[- \frac{\partial}{\partial k^{\beta}} \, F^{i} \left(\boldsymbol{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}), \dots \right) \; + \; \hat{L}^{i}_{j} \, \frac{\partial}{\partial k^{\beta}} \, \Phi^{j}(\boldsymbol{\theta} + \boldsymbol{\theta}_{0}, \mathbf{U}) \right] \, \theta^{\beta}_{0X}$$

The derivatives $\partial F^{i} / \partial \omega^{\beta}$ and $\partial F^{i} / \partial k^{\beta}$ are identically zero on Λ according to (1.3). We have then

$$\int_{0}^{2\pi} \!\! \dots \int_{0}^{2\pi} \kappa^{(q)}_{[\mathbf{U}(X,T)]\,i}(\boldsymbol{\theta} \, + \, \boldsymbol{\theta}_{0}(X,T)) \; f'^{i}_{(1)}(\boldsymbol{\theta},X,T) \, \frac{d^{m}\theta}{(2\pi)^{m}} \;\; \equiv \;\; 0$$

since all $\boldsymbol{\kappa}^{(q)}(\boldsymbol{\theta},X,T)$ are the left eigen-vectors of $\hat{L}$ with zero eigen-values.

It is easy to see also that all $\boldsymbol{\theta}_{0}(X,T)$ in the arguments of $\boldsymbol{\Phi}$ and $\boldsymbol{\kappa}^{(q)}$ will disappear after the integration w.r.t. $\boldsymbol{\theta}$ so we get the statement of the Lemma.

Lemma 1.1 is proved.

The system

$$\frac{\partial k^{\alpha}}{\partial U^{\nu}} \, U^{\nu}_{T} \;\; = \;\; \frac{\partial \omega^{\alpha}}{\partial U^{\nu}} \, U^{\nu}_{X} \;\;\; , \quad \alpha \; = \; 1, \dots, m$$

$$C^{(q)}_{\nu}(\mathbf{U}) \, U^{\nu}_{T} \;\; = \;\; D^{(q)}_{\nu}(\mathbf{U}) \, U^{\nu}_{X} \;\;\; , \quad q \; = \; 1, \dots, m+s \qquad (1.14)$$

is called the Whitham system for the m-phase solutions of system (1.1).

Let us note that we have $rank||\partial k^{\alpha} / \partial U^{\nu}|| = m$ according to our assumption above. In the generic case the derivatives U^{ν}_{T} can be expressed through U^{μ}_{X} and the Whitham system (1.14) can be written in the form

$$U^{\nu}_{T} \;=\; V^{\nu}_{\mu}(\mathbf{U})\, U^{\mu}_{X} \quad , \quad \nu,\, \mu \;=\; 1, \dots, N \tag{1.15}$$

where $V^{\nu}_{\mu}(U)$ is some $N \times N$ matrix depending on the variables $U^1, \dots, U^N$.

Let us say that quite often the system (1.1) can be written in the evolution form

$$\varphi^i_t \;\;=\;\; Q^i(\boldsymbol{\varphi},\, \boldsymbol{\varphi}_x,\, \boldsymbol{\varphi}_{xx},\, \dots\,) \tag{1.16}$$

For systems (1.16) the form (1.15) of the corresponding Whitham system has then a natural motivation.

We will assume here that if the conditions (1.14) are satisfied then the system (1.9) is resolvable on the space of 2π-periodic w.r.t. each θ^{α} functions. The solution $\mathbf{\Psi}_{(1)}(\boldsymbol{\theta}, X, T)$ is defined then modulo a linear combination of the "right eigen-functions" of $\hat{L}^i_{(X,T)j}$ $(\mathbf{\Psi}_{(0)\theta^{\alpha}},\, \mathbf{\Psi}_{(0)n^l})$ introduced above. According to common approach ([4, 12]) we can try to use the corresponding coefficients to make the systems analogous to (1.9) resolvable in the next orders and try to find recursively all the terms of series (1.5).

Different aspects and numerous applications of the Whitham method were studied in many different works ([1]-[50])[1] and the Whitham method is considered now as one of the classical methods of investigation of non-linear systems.

It was pointed out by G. Whitham ([1, 2, 3]) that the Whitham system (1.14) has a local Lagrangian structure in case when the initial system has a local Lagrangian structure

$$\delta \int\!\!\int \mathcal{L}(\boldsymbol{\varphi},\, \boldsymbol{\varphi}_t,\, \boldsymbol{\varphi}_x,\, \dots)\; dx\, dt \;\;=\;\; 0$$

on the space $\{\boldsymbol{\varphi}(x,t)\}$.

The procedure of construction of Lagrangian formalism for the Whitham system (1.14) is given by the averaging of the Lagrangian function $\mathcal{L}$ on the family of m-phase solutions of system (1.1) ([1, 2, 3]). Let us note also that in the case of presence of additional parameters n^l the additional method of Whitham pseudo-phases should be used.

The important procedure of averaging of local field-theoretical Hamiltonian structures was suggested by B.A. Dubrovin and S.P. Novikov ([14, 18, 27, 31]). The Dubrovin-Novikov procedure gives the local field-theoretical Hamiltonian formalism for the Whitham system (1.15) in the case when the initial system (1.16) has a local Hamiltonian formalism of general type. The Dubrovin-Novikov bracket for the Whitham system has a general form

$$\{U^{\nu}(X),\, U^{\mu}(Y)\} \;\;=\;\; g^{\nu\mu}(\mathbf{U})\;\delta'(X-Y) \;+\; b^{\nu\mu}_{\lambda}(\mathbf{U})\, U^{\lambda}_{X}\;\delta(X-Y) \tag{1.17}$$

and was called the local Poisson bracket of Hydrodynamic type. The theory of the brackets (1.17) is closely related with differential geometry ([14, 27, 31]) and is connected with different coordinate systems in the (pseudo) Euclidean spaces. Let us say also that during

[1] We apologize for the impossibility to give here the full list of numerous works on Whitham method.

the last years the important weakly-nonlocal generalizations of Dubrovin-Novikov brackets (Mokhov-Ferapontov bracket and Ferapontov brackets) were introduced and studied ([52, 53, 54, 55, 56, 57, 58]).

The Hamiltonian structure (1.17) for the systems (1.15) has a direct relation to the integrability of the systems of this class. Thus it was conjectured by S.P. Novikov that any diagonalizable system (1.15) Hamiltonian with respect to the bracket of Hydrodynamic Type is integrable. The conjecture of S.P. Novikov was proved by S.P. Tsarev ([51]) who suggested the "generalized Hodograph method" for solving the diagonal Hamiltonian systems (1.15). Let us say that the Tsarev method has become especially important for the Whitham systems corresponding to integrable hierarchies and provided a lot of very important solutions for such systems in different cases. We note here, that the Whitham systems of integrable hierarchies can usually be written in diagonal form ([3, 11, 25]) and admit the (multi-) Hamiltonian structures given by the averaging of the Lagrangian or the field-theoretical Hamiltonian structures of the initial system.

During the last years the theory of compatible Poisson brackets (1.17) and their deformations in connection with Quantum Field Theory was intensively developed ([59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71]). Namely, the theory of compatible Poisson brackets (1.17) plays the main role in the theory of "Frobenius manifolds" constructed by B.A. Dubrovin and connected with the classification of the Topological Quantum Field Theories (based on the Witten - Dijkgraaf - Verlinde - Verlinde equation). Every "Frobenius manifold" is connected also with the integrable hierarchy of Hydrodynamic Type

$$U^{\nu}_{t_s} \;\; = \;\; V^{\nu}_{(s)\mu}(\mathbf{U}) \, U^{\mu}_{X} \qquad (1.18)$$

having the bihamiltonian structure in Dubrovin-Novikov sense with a pair of Poisson brackets (1.17). [2]

The " ϵ-deformations " of the integrable hierarchies and the Poisson brackets (1.17) are connected with the "higher genus" corrections in the Quantum Field Theories and are being intensively investigated now ([63, 64, 65, 66, 67, 68, 69, 70, 71]). According to B.A. Dubrovin and Y. Zhang the ϵ-deformation of Frobenius manifold is given by the infinite series polynomial w.r.t. derivatives $\mathbf{U}_X$, $\mathbf{U}_{XX}$, $\mathbf{U}_{XXX}$, $\dots$ and representing the "small dispersion" deformation of the hierarchy (1.18) and the corresponding bihamiltonian structure of Hydrodynamic Type. The deformed hierarchy and bihamiltonian structure should possess also the additional properties of Frobenius manifolds ("unit vector field", Euler field, τ-symmetry etc. ...) and give then the "deformation" of the corresponding Frobenius manifold ([63, 65, 67]). The general problem of classification of deformations of bihamiltonian hierarchies (1.18) is being also intensively studied by now and very important results were obtained recently in this area ([66, 68, 69, 70, 71]).

The dispersive corrections to the Whitham systems were first considered by M.Y. Ablowitz and D.J. Benney ([5], also [6]-[7]) where the first consideration of multi-phase

[2] The hierarchy (1.18) and the corresponding pair of Poisson brackets possess also some additional properties (the existence of "unit vector field", Euler field, τ-symmetry etc. ...).

Whitham method was also made. As was shown in [5] the dispersive corrections to Whitham systems can naturally arise and, besides that, can be generalized to the multi-phase situation.

In this paper we will consider the deformations of the Whitham systems (1.15) having Dubrovin-Zhang form, i.e. the dispersive corrections to (1.15) containing the higher X-derivatives of the parameters $\mathbf{U}$. The problem in this form is connected with the problem set by B.A. Dubrovin which is formulated as the problem of deformations of Frobenius manifolds corresponding to the Whitham systems of integrable hierarchies. B.A. Dubrovin problem contains both the problem of deformation of Whitham systems (1.15) and the corresponding bi-Hamiltonian structures of Hydrodynamic Type giving the dispersive corrections to these structures.

We will consider here the first part of B.A. Dubrovin problem and suggest a general scheme of recursive construction of terms of the deformation of the Whitham system (1.15) using the initial system (1.1) or (1.16). The second part of B.A. Dubrovin problem will then be considered in the next paper. We will not require, however, the "integrability" of the system (1.15) and all the considerations below can be applicable for both integrable and non-integrable systems (1.1), (1.16). Thus, we do not require also the bihamiltonian property of the Whitham system (1.15).

In the next chapter we will make more detailed investigation of the asymptotic series (1.5) and describe the construction of the deformation procedure for general Whitham systems.

2 The deformation of the Whitham systems.

Let us start again with the description of the construction of asymptotic series (1.5) connected with the Whitham method. We will assume that we have the initial system having the general form (1.1) which has an N-parametric (modulo the initial phases θ_0^α) family of m-phase solutions

$$\varphi^i(x,t) \;=\; \Phi^i(\mathbf{k}(\mathbf{U})x \;+\; \boldsymbol{\omega}(\mathbf{U})t \;+\; \boldsymbol{\theta}_0,\, \mathbf{U}) \tag{2.1}$$

with the functions $\mathbf{\Phi}(\boldsymbol{\theta},\, \mathbf{U})$ satisfying the system

$$F^i\left(\mathbf{\Phi},\, \omega^\alpha(\mathbf{U})\,\mathbf{\Phi}_{\theta^\alpha},\, k^\beta(\mathbf{U})\,\mathbf{\Phi}_{\theta^\beta},\, \omega^\gamma(\mathbf{U})\,\omega^\delta(\mathbf{U})\,\mathbf{\Phi}_{\theta^\gamma\theta^\delta},\, \dots\right) \quad = \quad 0 \tag{2.2}$$

(and 2π-periodic w.r.t. all θ^α).

The choice of the functions $\Phi^i(\boldsymbol{\theta},\, \mathbf{U})$ at each $\mathbf{U}$ is defined modulo the initial phase $\boldsymbol{\theta}_0$ and the full family of solutions of (2.2) is given by all the functions $\mathbf{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_0,\, \mathbf{U})$ with arbitrary $\boldsymbol{\theta}_0 \;=\; (\theta_0^1, \dots, \theta_0^m)$. We will just assume here that the choice of $\Phi^i(\boldsymbol{\theta},\, \mathbf{U})$ is smooth on the family of m-phase solutions Λ.

We will assume also that the parameters $(\mathbf{k},\, \boldsymbol{\omega})$ are independent on the family Λ such that $N \geq 2m$. It will be convenient to use the parameters $(k^1, \dots, k^m, \omega^1, \dots, \omega^m, n^1, \dots, n^s)$ on the family Λ where k^α are the "wave numbers", ω^α are frequencies, and $(n^1, \dots, n^s)$

are additional parameters (if they present). The linearly independent solutions of the linearized system (2.2) will then be given by $m+s$ functions $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta},\, \mathbf{U})$, $\mathbf{\Phi}_{n^{l}}(\boldsymbol{\theta},\, \mathbf{U})$.

As we already said above we assume that the family Λ represents the "full regular family" of m-phase solutions of (2.3) such that all the requirements (1)-(3) formulated in the Definition 1.1 are satisfied. We have thus exactly $m+s$ functions $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta},\, \mathbf{U})$, $\mathbf{\Phi}_{n^{l}}(\boldsymbol{\theta},\, \mathbf{U})$ giving the basis in the kernel of linearized system (2.2) and exactly $m+s$ linearly independent functions $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_{0})$ giving the basis in the kernel of the adjoint operator (for generic $\mathbf{k}$, $\boldsymbol{\omega}$) and depending in a smooth way on the parameters $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$.

As we mentioned already we make a rescaling of coordinates $X = \epsilon\, x$, $T = \epsilon\, t$ and try to find the solutions of system

$$F^{i}(\boldsymbol{\varphi},\, \epsilon\, \boldsymbol{\varphi}_{T},\, \epsilon\, \boldsymbol{\varphi}_{X},\, \epsilon^{2}\, \boldsymbol{\varphi}_{TT}, \epsilon^{2}\, \boldsymbol{\varphi}_{XT}, \epsilon^{2}\, \boldsymbol{\varphi}_{XX}, \dots) \;=\; 0 \quad , \quad i=1,\dots,n \tag{2.3}$$

having the form

$$\phi^{i}(\boldsymbol{\theta}, X, T, \epsilon) \;=\; \Psi^{i}\left(\frac{\mathbf{S}(X,T)}{\epsilon} \;+\; \boldsymbol{\theta}, X, T, \epsilon\right) \tag{2.4}$$

$$\Psi^{i}(\boldsymbol{\theta}, X, T, \epsilon) \;=\; \sum_{k\geq 0} \Psi^{i}_{(k)}(\boldsymbol{\theta}, X, T)\; \epsilon^{k} \tag{2.5}$$

The function $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta}, X, T)$ belongs to the family Λ at every fixed X and T and the compatibility conditions (1.12) for the system

$$\hat{L}^{i}_{j}\, \Psi^{j}_{(1)}(\boldsymbol{\theta}, X, T) \;=\; f^{i}_{(1)}(\boldsymbol{\theta}, X, T) \tag{2.6}$$

give the Whitham system (1.13) on the parameters $\mathbf{U}(X,T)$ of the zero approximation. The function $\mathbf{\Psi}_{(1)}(\boldsymbol{\theta}, X, T)$ is defined from the system (2.6) modulo the linear combination of functions $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta},\, \mathbf{U})$, $\mathbf{\Phi}_{n^{l}}(\boldsymbol{\theta},\, \mathbf{U})$ at every X and T.

All the other approximations $\mathbf{\Psi}_{(k)}(\boldsymbol{\theta}, X, T)$ satisfy the linear systems

$$\hat{L}^{i}_{j}\, \Psi^{j}_{(k)}(\boldsymbol{\theta}, X, T) \;=\; f^{i}_{(k)}(\boldsymbol{\theta}, X, T) \tag{2.7}$$

where the functions $\mathbf{f}_{(k)}(\boldsymbol{\theta}, X, T)$ represent the higher order discrepancies given by system (2.3).

The compatibility conditions of the systems (2.7) ($k \geq 2$) give the restrictions on the "initial phases" θ^{α}_{0} of the zero approximation and on the previous corrections $\mathbf{\Psi}_{(k')}(\boldsymbol{\theta}, X, T)$.

Let us make now one more general assumption about the systems (2.6) and (2.7). Namely we omit here the special investigation of the solvability of systems (2.6), (2.7) on the space of 2π-periodic functions and assume that the orthogonality conditions

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi} \kappa^{(q)}_{[\mathbf{U}(X,T)]\, i}(\boldsymbol{\theta} \,+\, \boldsymbol{\theta}_{0}(X,T))\;\; f^{i}_{(k)}(\boldsymbol{\theta}, X, T)\;\; \frac{d^{m}\theta}{(2\pi)^{m}} \;\;=\;\; 0 \tag{2.8}$$

give the necessary and sufficient conditions of solvability of these systems. So, we will assume that under the conditions (2.8) we can always find a smooth 2π-periodic solution of (2.6), (2.7) defined modulo the linear combination

$$\sum_{\alpha=1}^{m} c^{\alpha}_{(k)}(X,T)\ \ \mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta},\,X,\,T)\ \ +\ \ \sum_{l=1}^{s} d^{l}_{(k)}(X,T)\ \ \mathbf{\Phi}_{n^{l}}(\boldsymbol{\theta},\,X,\,T) \tag{2.9}$$

Let us consider now the construction of the asymptotic series (2.5). We note first that the solution $\mathbf{\Psi}_{(k)}(\boldsymbol{\theta},X,T)$ is defined from the system (2.7) modulo the linear combination (2.9) which is equivalent in the main order to the addition of the values $\epsilon^{k+1}\, c^{\alpha}_{(k)}(X,T)$ to the phases $S^{\alpha}(X,T)$ of the zero approximation $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta},X,T)$ and to the addition of the values $\epsilon^{k}\, d^{l}_{(k)}(X,T)$ to the parameters $n^{l}(X,T)$ of $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta},X,T)$ according to the formulae (2.4), (2.5). Easy to see also that we can add the values $\epsilon^{k+1}\, c^{\alpha}_{(k)X}(X,T)$ and $\epsilon^{k+1}\, c^{\alpha}_{(k)T}(X,T)$ to the parameters k^{α} and ω^{α} in the $\mathbf{U}$-dependence of $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta},X,T)$ which does not affect the k-th order of ϵ in the series (2.5).

Let us change now the procedure of construction of series (2.5) in the following way:

1) At every step k we choose the solution $\mathbf{\Psi}_{(k)}(\boldsymbol{\theta},X,T)$ in arbitrary way;

2) We allow the regular ϵ-dependence

$$S^{\alpha}(X,T,\epsilon)\ \ =\ \ \sum_{k\geq 0} S^{\alpha}_{(k)}(X,T)\ \epsilon^{k}$$

$$n^{l}(X,T,\epsilon)\ \ =\ \ \sum_{k\geq 0} n^{l}_{(k)}(X,T)\ \epsilon^{k}$$

of the phases S^{α} and the parameters $(\mathbf{k},\boldsymbol{\omega},\mathbf{n})$ of the zero approximation $\mathbf{\Psi}_{(0)}(\boldsymbol{\theta},X,T)$ such that

$$k^{\alpha}(X,T,\epsilon)\ \ =\ \ S^{\alpha}_{X}(X,T,\epsilon)\ \ \ ,\ \ \ \omega^{\alpha}(X,T,\epsilon)\ \ =\ \ S^{\alpha}_{T}(X,T,\epsilon)$$

3) We use the higher orders $S^{\alpha}_{(k)}$, $n^{l}_{(k)}$ of the functions $S^{\alpha}(X,T,\epsilon)$, $n^{l}(X,T,\epsilon)$ to provide the orthogonality conditions (2.8) for the systems (2.7).

We can write now the asymptotic solution of (2.3) in the form

$$\phi^{i}(\boldsymbol{\theta},X,T,\epsilon)\ \ =\ \ \Phi^{i}\left(\frac{\mathbf{S}(X,T,\epsilon)}{\epsilon}+\boldsymbol{\theta},\ \ \mathbf{S}_{X}(X,T,\epsilon),\ \ \mathbf{S}_{T}(X,T,\epsilon),\ \ \mathbf{n}(X,T,\epsilon)\right)\ +$$
$$+\ \sum_{k\geq 1}\tilde{\Psi}^{i}_{(k)}\left(\frac{\mathbf{S}(X,T,\epsilon)}{\epsilon}+\boldsymbol{\theta},\ X,\ T\right)\ \epsilon^{k} \tag{2.10}$$

Let us note that the phase shift $\boldsymbol{\theta}_{0}(X,T)$ of the initial approximation becomes now the ϵ-term in ϵ-expansion of the function $\mathbf{S}(X,T,\epsilon)$.

It's not difficult to see that the series (2.10) can be always represented in the form (2.4)-(2.5) and give the same set of the asymptotic solutions of system (2.3). However, the series (2.10) contains a big "renormalization freedom" since we can change in arbitrary way the higher orders corrections $\mathbf{S}_{(2)}(X,T)$, $\mathbf{S}_{(3)}(X,T)$, $\dots$, $\mathbf{n}_{(1)}(X,T)$, $\mathbf{n}_{(2)}(X,T)$, $\dots$ and then adjust the functions $\tilde{\mathbf{\Psi}}_{(k)}$ in the appropriate way.

Let us now "fix the normalization" of the solution (2.10) in the following way:

We require that the first term

$$\Phi^{i}\left(\frac{\mathbf{S}(X,T,\epsilon)}{\epsilon}+\boldsymbol{\theta},\ \ \mathbf{S}_{X}(X,T,\epsilon),\ \ \mathbf{S}_{T}(X,T,\epsilon),\ \ \mathbf{n}(X,T,\epsilon)\right)$$

of (2.10) gives "the best approximation" of the asymptotic solution (2.10) by the modulated m-phase solution of (1.1). To be more precise, we require that the rest of the series (2.10) is orthogonal (at every X and T) to the vectors $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta},\,X,\,T)$, $\mathbf{\Phi}_{n^{l}}(\boldsymbol{\theta},\,X,\,T)$ "tangent" to Λ at the "point" $[\mathbf{S}(X,T,\epsilon),\mathbf{n}(X,T,\epsilon)]$.

Thus we will put now the conditions

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi}\sum_{i=1}^{n}\Phi^{i}_{\theta^{\alpha}}(\boldsymbol{\theta},\,\mathbf{S}_{X},\,\mathbf{S}_{T},\,\mathbf{n})\ \left[\sum_{k=1}^{\infty}\tilde{\Psi}^{i}_{(k)}(\boldsymbol{\theta},\,X,\,T)\ \epsilon^{k}\right]\ \frac{d^{m}\theta}{(2\pi)^{m}}\ \ =\ \ 0 \qquad (2.11)$$

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi}\sum_{i=1}^{n}\Phi^{i}_{n^{l}}(\boldsymbol{\theta},\,\mathbf{S}_{X},\,\mathbf{S}_{T},\,\mathbf{n})\ \left[\sum_{k=1}^{\infty}\tilde{\Psi}^{i}_{(k)}(\boldsymbol{\theta},\,X,\,T)\ \epsilon^{k}\right]\ \frac{d^{m}\theta}{(2\pi)^{m}}\ \ =\ \ 0 \qquad (2.12)$$

(for all ϵ).

As we saw above, the functions $\tilde{\mathbf{\Psi}}_{(k)}(\boldsymbol{\theta},X,T)$ are defined modulo the linear combinations of the functions $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta},\,\mathbf{S}_{(0)X},\,\mathbf{S}_{(0)T},\,\mathbf{n}_{(0)})$, $\mathbf{\Phi}_{n^{l}}(\boldsymbol{\theta},\,\mathbf{S}_{(0)X},\,\mathbf{S}_{(0)T},\,\mathbf{n}_{(0)})$ which are linearly independent for the full regular family of m-phase solutions of (1.1). Using this fact it's not difficult to prove that the conditions (2.11)-(2.12) fix uniquely all the terms $\mathbf{S}_{(k)}$, $\mathbf{n}_{(k)}$, and $\tilde{\mathbf{\Psi}}_{(k)}(\boldsymbol{\theta},X,T)$ $(k\geq 0)$ for a given solution (2.10).

Let us say now that the choice of the normalization (2.11)-(2.12) is not unique. In particular, it depends on the choice of the variables $\varphi^{i}(x,t)$ for the "vector" $(i>1)$ systems (1.1). We will speak about the normalization (2.11)-(2.12) as about one possible way to fix the functions $\tilde{\mathbf{\Psi}}_{(k)}(\boldsymbol{\theta},X,T)$.

The solution $\boldsymbol{\phi}\ (\boldsymbol{\theta},X,T)$ can now be defined as the asymptotic solution of the system (2.3) having the form (2.10) which satisfies the conditions

$$\int_{0}^{2\pi}\!\!\dots\int_{0}^{2\pi}\sum_{i=1}^{n}\Phi^{i}_{\theta^{\alpha}}(\boldsymbol{\theta},\,\mathbf{S}_{X},\,\mathbf{S}_{T},\,\mathbf{n})\ \ \phi^{i}\left(\boldsymbol{\theta}-\frac{\mathbf{S}}{\epsilon},\,X,\,T,\,\epsilon\right)\ \frac{d^{m}\theta}{(2\pi)^{m}}\ \ =\ \ 0$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^{n} \Phi^i_{n^l}(\boldsymbol{\theta}, \mathbf{S}_X, \mathbf{S}_T, \mathbf{n}) \left[\phi^i \left(\boldsymbol{\theta} - \frac{\mathbf{S}}{\epsilon}, \, X, T, \epsilon \right) - \Phi^i(\boldsymbol{\theta}, \mathbf{S}_X, \mathbf{S}_T, \mathbf{n}) \right] \frac{d^m\theta}{(2\pi)^m} = 0$$

(for all ϵ).

To find a solution in the form (2.10) we substitute the series (2.10) in the system (2.3) and try to find the functions $\tilde{\Psi}^i_{(k)}(\boldsymbol{\theta}, X, T)$ from the linear systems

$$\hat{L}^i_{j[\mathbf{S}_{(0)},\mathbf{n}_{(0)}]} \; \tilde{\Psi}^j_{(k)}(\boldsymbol{\theta}, X, T) \;\; = \;\; \tilde{f}^i_{(k)}(\boldsymbol{\theta}, X, T) \tag{2.13}$$

analogous to (2.7).

The functions $\tilde{f}^i_{(k)}(\boldsymbol{\theta}, X, T)$ are different from the functions $f^i_{(k)}(\boldsymbol{\theta}, X, T)$ since we included "a part of ϵ-dependence" in the first term of (2.10). The functions $\tilde{\mathbf{f}}_{(k)}(\boldsymbol{\theta}, X, T)$ should be orthogonal to the functions $\kappa^{(q)}_{[\mathbf{S}_{(0)},\mathbf{n}_{(0)}]}(\boldsymbol{\theta}, X, T)$, $q = 1, \dots, m+s$ and we use the conditions (2.11)-(2.12) for the recurrent determination of $\tilde{\boldsymbol{\Psi}}_{(k)}(\boldsymbol{\theta}, X, T)$. The functions $S^\alpha_{(k)}(X,T)$, $n^l_{(k)}(X,T)$ are used now to provide the resolvability of the systems (2.13) in all orders of ϵ.

Let us investigate now the systems arising on the functions $\mathbf{S}_{(0)}(X,T)$, $\mathbf{S}_{(1)}(X,T)$, $\dots$, $\mathbf{n}_{(0)}(X,T)$, $\mathbf{n}_{(1)}(X,T)$, $\dots$. Let us note that we have in our notations $\mathbf{S}(X,T) = \mathbf{S}_{(0)}(X,T)$, $\boldsymbol{\theta}_0(X,T) = \mathbf{S}_{(1)}(X,T)$. We saw in Lemma 1.1 that the function $\boldsymbol{\theta}_0(X,T)$ does not appear in the solvability conditions of the system (1.9). For the asymptotic solution written in the form (2.10) with the normalization conditions (2.11)-(2.12) we can prove here even stronger statement.

Lemma 2.1

The functions $S^\alpha_{(k)}(X,T)$, $n^l_{(k)}(X,T)$ do not appear in the expression for the discrepancy $\tilde{\mathbf{f}}_{(k)}(\boldsymbol{\theta}, X, T)$ and do not affect the solution $\tilde{\boldsymbol{\Psi}}_{(k)}(\boldsymbol{\theta}, X, T)$.

Proof.

The way how we get the discrepancy $\tilde{\mathbf{f}}_{(k)}(\boldsymbol{\theta}, X, T)$ can be described as follows:

1) We substitute the solution (2.10) in the system (2.3);

2) After "making all differentiations" we can omit the argument shift $\mathbf{S}(X,T,\epsilon)/\epsilon$ in all functions depending on $\boldsymbol{\theta}$;

3) Then collecting together all the terms of order ϵ^k we get the system (2.13).

It's not difficult to see that in this approach the k-th order of ϵ containing the functions $\mathbf{S}_{(k)}(X,T)$ or $\mathbf{n}_{(k)}(X,T)$ have the form

$$\frac{dF^i}{dk^\alpha} \, S^\alpha_{(k)X} \quad , \quad \frac{dF^i}{d\omega^\alpha} \, S^\alpha_{(k)T} \quad , \quad \text{and} \quad \frac{dF^i}{dn^l} \, n^l_{(k)}$$

where F^i are the constraints (2.2) defining the m-phase solutions of (1.1). The derivatives d/dk^α, $d/d\omega^\alpha$, d/dn^l here are the total derivatives including the dependence of the functions $\Phi^i(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ on the corresponding parameters view the form of (2.10). However, all

these derivatives are identically equal to zero on the family Λ, so we get the first part of the Lemma. To finish the proof we have to note also that the values $\mathbf{S}_{(k)}(X,T)$, $\mathbf{n}_{(k)}(X,T)$ do not appear in the k-th order of ϵ of the normalization conditions (2.11)-(2.12) either.

Lemma 2.1 is proved.

We can formulate now the procedure in the following form:

We try to solve the systems (2.13) recursively in all orders and find the functions $\tilde{\mathbf{\Psi}}_{(k)}(\boldsymbol{\theta},X,T)$ satisfying the conditions (2.11)-(2.12). At each k-th step of our procedure we get a system on the functions $\mathbf{S}_{(k-1)}(X,T)$, $\mathbf{n}_{(k-1)}(X,T)$ from the solvability conditions of (2.13). The full set of Whitham solutions will then be parameterized by the set of all functions $\{\mathbf{S}_{(k)}(X,T), \mathbf{n}_{(k)}(X,T)\}$, $k \geq 0$ satisfying the conditions of solvability of systems (2.13).

Now let us investigate the systems arising on the functions $\mathbf{S}_{(k-1)}(X,T)$, $\mathbf{n}_{(k-1)}(X,T)$ from the solvability conditions of (2.13)

$$\int_0^{2\pi} \dots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{S}_0,\mathbf{n}_0]i}(\boldsymbol{\theta},X,T) \; \tilde{f}^i_{(k)}(\boldsymbol{\theta},X,T) \; \frac{d^m\theta}{(2\pi)^m} \quad \equiv \quad 0 \quad , \qquad q=1,\dots,m+s \quad (2.14)$$

in the k-th order of ϵ. We note first that the solvability conditions of (2.13) for $k=1$ give a nonlinear system on the functions $\mathbf{S}_{(0)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$. It's not difficult to prove the following Lemma:

Lemma 2.2.

The system arising for $k=1$ coincides with the Whitham system for the functions $\mathbf{S}_{(0)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$.

Proof.

Indeed, using Lemma 2.1 it's not difficult to see that the discrepancy $\mathbf{f}_{(1)}(\boldsymbol{\theta},X,T)$ differs from $\tilde{\mathbf{f}}_{(1)}(\boldsymbol{\theta},X,T)$ just by the terms containing derivatives $\boldsymbol{\theta}_{0X}$ and $\boldsymbol{\theta}_{0T}$. However, according to Lemma 1.1 these terms do not affect the orthogonality conditions (1.12) which then coincide with orthogonality conditions for $\tilde{\mathbf{f}}_{(1)}(\boldsymbol{\theta},X,T)$.

Lemma 2.2 is proved.

The Whitham system (orthogonality conditions) for the functions $\mathbf{S}_{(0)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$ can be written in the following general form:

$$W^{(q)}\left[\mathbf{S}_{(0)},\mathbf{n}_{(0)}\right] \quad \equiv \quad A^{(q)}_{\alpha}\left(\mathbf{S}_{(0)X},\mathbf{S}_{(0)T},\mathbf{n}_{(0)}\right) \; S^{\alpha}_{(0)TT} \; +$$
$$+ \; B^{(q)}_{\alpha}\left(\mathbf{S}_{(0)X},\mathbf{S}_{(0)T},\mathbf{n}_{(0)}\right) \; S^{\alpha}_{(0)XT} \; + \; C^{(q)}_{\alpha}\left(\mathbf{S}_{(0)X},\mathbf{S}_{(0)T},\mathbf{n}_{(0)}\right) \; S^{\alpha}_{(0)XX} \; +$$
$$+ \; G^{(q)}_{l}\left(\mathbf{S}_{(0)X},\mathbf{S}_{(0)T},\mathbf{n}_{(0)}\right) \; n^{l}_{(0)T} \; + \; H^{(q)}_{l}\left(\mathbf{S}_{(0)X},\mathbf{S}_{(0)T},\mathbf{n}_{(0)}\right) \; n^{l}_{(0)X} \; = \; 0 \qquad (2.15)$$

with some functions $A^{(q)}_{\alpha}$, $B^{(q)}_{\alpha}$, $C^{(q)}_{\alpha}$, $G^{(q)}_{l}$, $H^{(q)}_{l}$, $\quad q = 1, \dots, m+s, \quad \alpha = 1, \dots, m$, $l = 1, \dots, s$.

Let us prove now the following Lemma about the systems on $\mathbf{S}_{(k)}$, $\mathbf{n}_{(k)}$, $\quad k \geq 1$.

Lemma 2.3.

The orthogonality conditions of $\tilde{\mathbf{f}}_{(k+1)}(\boldsymbol{\theta}, X, T)$ to the functions $\boldsymbol{\kappa}^{(q)}_{[\mathbf{S}_{(0)},\mathbf{n}_{(0)}]}$ give the following linear systems for the functions $\mathbf{S}_{(k)}(X,T)$, $\mathbf{n}_{(k)}(X,T)$, $\quad k \geq 1$:

$$\int\!\!\int \frac{\delta W^{(q)}(X,T)}{\delta S^{\alpha}_{(0)}(X',T')}\, S^{\alpha}_{(k)}(X',T')\, dX'\, dT' \; + \; \int\!\!\int \frac{\delta W^{(q)}(X,T)}{\delta n^{l}_{(0)}(X',T')}\, n^{l}_{(k)}(X',T')\, dX'\, dT' \; =$$

$$= \; V^{(q)}_{(k)} \left[\mathbf{S}_{(0)}, \dots, \mathbf{S}_{(k-1)}, \mathbf{n}_{(0)}, \dots, \mathbf{n}_{(k-1)} \right] (X,T)$$

In other words, the functions $\mathbf{S}_{(k)}(X,T)$, $\mathbf{n}_{(k)}(X,T)$ satisfy the linearized Whitham system on the functions $\mathbf{S}_{(0)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$ with additional right-hand part depending on the functions $\mathbf{S}_{(0)}$, ..., $\mathbf{S}_{(k-1)}$, $\mathbf{n}_{(0)}$, ..., $\mathbf{n}_{(k-1)}$.

Proof.

Let us look at the terms in $\tilde{\mathbf{f}}_{(k+1)}(\boldsymbol{\theta}, X, T)$ which contain the functions $\mathbf{S}_{(k)}(X,T)$, $\mathbf{n}_{(k)}(X,T)$:

1) As we proved in Lemma 2.1 the functions $\tilde{f}^{i}_{(1)}(\boldsymbol{\theta}, X, T)$ contain only the terms depending on $\mathbf{S}_{(0)}$, $\mathbf{n}_{(0)}$. Easy to see that the functions $\tilde{f}^{i}_{(k+1)}(\boldsymbol{\theta}, X, T)$ will then contain the terms

$$\int\!\!\int \frac{\delta \tilde{f}^{i}_{(1)}(\boldsymbol{\theta}, X, T)}{\delta S^{\alpha}_{(0)}(X',T')}\, S^{\alpha}_{(k)}(X',T')\, dX'\, dT' \; + \; \int\!\!\int \frac{\delta \tilde{f}^{i}_{(1)}(\boldsymbol{\theta}, X, T)}{\delta n^{l}_{(0)}(X',T')}\, n^{l}_{(k)}(X',T')\, dX'\, dT'$$

according to the form of the first term in (2.10).

2) There are terms containing the functions $\mathbf{S}_{(k)}$, $\mathbf{n}_{(k)}$ and the function $\tilde{\mathbf{\Psi}}_{(1)}(\boldsymbol{\theta}, X, T)$. All such terms can be written in the form:

$$-\int\!\!\int \left[S^{\alpha}_{(k)}(X',T')\, \frac{\delta \hat{L}^{i}_{[\mathbf{S}_0,\mathbf{n}_0]j}(X,T)}{\delta S^{\alpha}_{(0)}(X',T')} \, + \, n^{l}_{(k)}(X',T')\, \frac{\delta \hat{L}^{i}_{[\mathbf{S}_0,\mathbf{n}_0]j}(X,T)}{\delta n^{l}_{(0)}(X',T')} \right] \tilde{\Psi}^{j}_{(1)}(\boldsymbol{\theta}, X, T)\, dX'\, dT'$$

where $\hat{L}^{i}_{j[\mathbf{S}_0,\mathbf{n}_0]}$ is the linear operator (1.10) given by the linearization of the system (2.2) on the family of m-phase solutions.

3) There will the terms of the form

$$- \int \dots \int \frac{\delta^2 F^i_{[\mathbf{S}_0,\mathbf{n}_0]}(\boldsymbol{\theta}, X, T)}{\delta S^{\alpha}_{(0)}(X', T')\, \delta S^{\beta}_{(0)}(X'', T'')} \; S^{\alpha}_{(k)}(X', T') \; S^{\beta}_{(1)}(X'', T'') \; dX' \, dT' \, dX'' \, dT'' \; - \; \dots$$

where $F^i_{[\mathbf{S}_0,\mathbf{n}_0]}(\boldsymbol{\theta}, X, T)$ is the left-hand part of the system (2.2).

However, the sum of all such terms is equal to zero since they all correspond to the expansion of the "shift of parameters" $\mathbf{S}_{(0)}(X, T)$, $\mathbf{n}_{(0)}(X, T)$ on the family Λ where we have $F^i(\boldsymbol{\theta}, X, T) \equiv 0$ identically.

Let us look now at the terms in the orthogonality conditions

$$\int_0^{2\pi} \dots \int_0^{2\pi} \; \kappa^{(q)}_{[\mathbf{S}_0,\mathbf{n}_0]i}(\boldsymbol{\theta}, X, T) \quad \tilde{f}^i_{(k+1)}(\boldsymbol{\theta}, X, T) \quad \frac{d^m \theta}{(2\pi)^m} \quad = \quad 0 \tag{2.16}$$

containing the terms 1)-2).

We have identically

$$\int_0^{2\pi} \dots \int_0^{2\pi} \; \kappa^{(q)}_{[\mathbf{S}_0,\mathbf{n}_0]i}(\boldsymbol{\theta}, X, T) \quad L^i_{[\mathbf{S}_0,\mathbf{n}_0]j}(\boldsymbol{\theta}, \boldsymbol{\theta}', X, T) \quad \frac{d^m \theta}{(2\pi)^m} \quad \equiv \quad 0 \tag{2.17}$$

on Λ (where $L^i_{[\mathbf{S}_0,\mathbf{n}_0]j}(\boldsymbol{\theta}, \boldsymbol{\theta}', X, T)$ is the "core" of the operator $\hat{L}^i_{[\mathbf{S}_0,\mathbf{n}_0]j}(X, T)$).

Easy to see then that the inner product of $\boldsymbol{\kappa}^{(q)}$ with the terms 2) is equal to

$$\int_0^{2\pi} \!\! \dots \! \int_0^{2\pi} \!\! \iint \left[S^{\alpha}_{(k)}(X', T') \, \frac{\delta \kappa^{(q)}_{[\mathbf{S}_0,\mathbf{n}_0]i}(\boldsymbol{\theta}, X, T)}{\delta S^{\alpha}_{(0)}(X', T')} \; + \; n^l_{(k)}(X', T') \, \frac{\delta \kappa^{(q)}_{[\mathbf{S}_0,\mathbf{n}_0]i}(\boldsymbol{\theta}, X, T)}{\delta n^l_{(0)}(X', T')} \right] dX' dT' \; \times$$

$$\times \;\; \hat{L}^i_{[\mathbf{S}_0,\mathbf{n}_0]j}(X, T) \;\; \tilde{\Psi}^j_{(1)}(\boldsymbol{\theta}, X, T) \;\; \frac{d^m \theta}{(2\pi)^m}$$

It's not difficult to see now that the terms of orthogonality conditions containing the terms 1)-2) can be written together in the form

$$\iint S^{\alpha}_{(k)}(X', T') \; \frac{\delta}{\delta S^{\alpha}_{(0)}(X', T')} \; \langle \boldsymbol{\kappa}^{(q)}_{[\mathbf{S}_0,\mathbf{n}_0]} \; \cdot \; \tilde{\mathbf{f}}_{(1)}[\mathbf{S}_0, \mathbf{n}_0] \rangle (X, T) \;\; dX' \, dT' \; +$$

$$+ \;\; \iint n^l_{(k)}(X', T') \; \frac{\delta}{\delta n^l_{(0)}(X', T')} \; \langle \boldsymbol{\kappa}^{(q)}_{[\mathbf{S}_0,\mathbf{n}_0]} \; \cdot \; \tilde{\mathbf{f}}_{(1)}[\mathbf{S}_0, \mathbf{n}_0] \rangle (X, T) \;\; dX' \, dT'$$

(where $< \dots \cdot \dots >$ is the inner product of $\boldsymbol{\kappa}^{(q)}$ and $\tilde{\mathbf{f}}_{(1)}$).

All the other terms in the orthogonality conditions (2.16) are the smooth functionals of $\mathbf{S}_{(0)}$, …, $\mathbf{S}_{(k-1)}$, $\mathbf{n}_{(0)}$, …, $\mathbf{n}_{(k-1)}$, so we get the statement of the Lemma.

Lemma 2.3 is proved.

Let us consider now the systems (1.1) satisfying the special non-degeneracy conditions. Namely, we will assume that the corresponding Whitham system (2.15) can be resolved

w.r.t. to the highest T-derivatives of the functions $\mathbf{S}_{(0)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$ and written in the "evolution" form:

$$S^{\alpha}_{(0)TT} \;=\; M^{\alpha}_{(0)\beta}\left(\mathbf{S}_{(0)X}, \mathbf{S}_{(0)T}, \mathbf{n}_{(0)}\right)\; S^{\beta}_{(0)XX} \;+\; N^{\alpha}_{(0)\beta}\left(\mathbf{S}_{(0)X}, \mathbf{S}_{(0)T}, \mathbf{n}_{(0)}\right)\; S^{\beta}_{(0)TX} \;+$$
$$+\; P^{\alpha}_{(0)p}\left(\mathbf{S}_{(0)X}, \mathbf{S}_{(0)T}, \mathbf{n}_{(0)}\right)\; n^{p}_{(0)X} \quad , \qquad \alpha \,=\, 1,\dots,m \tag{2.18}$$

$$n^{l}_{(0)T} \;=\; T^{l}_{(0)\beta}\left(\mathbf{S}_{(0)X}, \mathbf{S}_{(0)T}, \mathbf{n}_{(0)}\right)\; S^{\beta}_{(0)XX} \;+\; L^{l}_{(0)\beta}\left(\mathbf{S}_{(0)X}, \mathbf{S}_{(0)T}, \mathbf{n}_{(0)}\right)\; S^{\beta}_{(0)TX} \;+$$
$$+\; R^{l}_{(0)p}\left(\mathbf{S}_{(0)X}, \mathbf{S}_{(0)T}, \mathbf{n}_{(0)}\right)\; n^{p}_{(0)X} \quad , \qquad l \,=\, 1,\dots,s \tag{2.19}$$

After the introduction of the variables $k^{\alpha}_{(0)} \,=\, S^{\alpha}_{(0)X}$, $\omega^{\alpha}_{(0)} \,=\, S^{\alpha}_{(0)T}$ we can write the system (2.18)-(2.19) in the form

$$k^{\alpha}_{(0)T} \;=\; \omega^{\alpha}_{(0)X}$$

$$\omega^{\alpha}_{(0)T} \;=\; M^{\alpha}_{(0)\beta}\left(\mathbf{k}_{(0)}, \boldsymbol{\omega}_{(0)}, \mathbf{n}_{(0)}\right)\; k^{\beta}_{(0)X} \;+\; N^{\alpha}_{(0)\beta}\left(\mathbf{k}_{(0)}, \boldsymbol{\omega}_{(0)}, \mathbf{n}_{(0)}\right)\; \omega^{\beta}_{(0)X} \;+$$
$$+\; P^{\alpha}_{(0)p}\left(\mathbf{k}_{(0)}, \boldsymbol{\omega}_{(0)}, \mathbf{n}_{(0)}\right)\; n^{p}_{(0)X} \tag{2.20}$$

$$n^{l}_{(0)T} \;=\; T^{l}_{(0)\beta}\left(\mathbf{k}_{(0)}, \boldsymbol{\omega}_{(0)}, \mathbf{n}_{(0)}\right)\; k^{\beta}_{(0)X} \;+\; L^{l}_{(0)\beta}\left(\mathbf{k}_{(0)}, \boldsymbol{\omega}_{(0)}, \mathbf{n}_{(0)}\right)\; \omega^{\beta}_{(0)X} \;+$$
$$+\; R^{l}_{(0)p}\left(\mathbf{k}_{(0)}, \boldsymbol{\omega}_{(0)}, \mathbf{n}_{(0)}\right)\; n^{p}_{(0)X}$$

i.e. in the form (1.15).

We will consider now the hyperbolic systems (2.20), i.e. such that the matrix

$$V^{\nu}_{\mu}(\mathbf{k}_{(0)}, \boldsymbol{\omega}_{(0)}, \mathbf{n}_{(0)}) \;=\; \left(\begin{array}{c|c|c} 0 & I_m & 0 \\ \hline M_{(0)} & N_{(0)} & P_{(0)} \\ \hline T_{(0)} & L_{(0)} & R_{(0)} \end{array}\right)$$

has exactly $N = 2m + s$ real eigen-values with N linearly independent real eigen-vectors. For hyperbolic systems (2.20) it's natural to consider the Cauchy problem with the smooth initial data $\mathbf{k}_{(0)}(X,0)$, $\boldsymbol{\omega}_{(0)}(X,0)$, $\mathbf{n}_{(0)}(X,0)$ (or $\mathbf{S}_{(0)}(X,0)$, $\mathbf{S}_{(0)T}(X,0)$, $\mathbf{n}_{(0)}(X,0)$). The smooth solution of (2.20) exists in general up to some finite time T_0 until the breakdown occurs. So we can write the zero (global in X) approximation for the solution (2.10) just in the time interval where we have a smooth solution of the Whitham system. Using Lemma 2.3 it's not difficult to prove then the following Lemma:

Lemma 2.4

For non-degenerate hyperbolic Whitham system (2.20) and the global solution $\mathbf{S}_{(0)}(X,T)$*,* $\mathbf{n}_{(0)}(X,T)$ *defined on the interval* $[0,T_0]$ *the higher orders approximations in (2.10) are all defined for all* X *and* $T \in [0,T_0)$ *and are parameterized by the initial values* $\mathbf{S}_{(k)}(X,0)$*,* $\mathbf{S}_{(k)T}(X,0)$*,* $\mathbf{n}_{(k)}(X,0)$*.* [3]

Proof.

Indeed, as follows from Lemma 2.3 the functions $\mathbf{S}_{(k)}(X,T)$, $\mathbf{n}_{(k)}(X,T)$ are defined by the initial values $\mathbf{S}_{(k)}(X,0)$, $\mathbf{S}_{(k)T}(X,0)$, $\mathbf{n}_{(k)}(X,0)$ and can be found from the linear system using the characteristic directions of (2.20) (defined by $\mathbf{S}_{(0)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$) provided that all the smooth solutions $\mathbf{S}_{(0)}(X,T)$, ... , $\mathbf{S}_{(k-1)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$ ... , $\mathbf{n}_{(k-1)}(X,T)$ and $\tilde{\mathbf{\Psi}}_{(1)}(\boldsymbol{\theta},X,T)$, ... , $\tilde{\mathbf{\Psi}}_{(k-2)}(\boldsymbol{\theta},X,T)$ exist on the interval $[0,T_0)$. According to Lemma 2.1 and Lemma 2.3 we can find then the functions $\tilde{\Psi}^i_{(k-1)}(\boldsymbol{\theta},X,T)$ which are the local expressions (in X and T) of $\mathbf{S}_{(0)}(X,T)$, ... , $\mathbf{S}_{(k-1)}(X,T)$, $\mathbf{n}_{(0)}(X,T)$ ... , $\mathbf{n}_{(k-1)}(X,T)$ and their derivatives. Using the induction we then finish the proof of the Lemma.

Lemma 2.4 is proved.

According to Lemma 2.4 we can formulate now the following statement:

For the initial system (1.1) having the non-degenerate hyperbolic Whitham system (2.20) the corresponding Whitham solutions (2.10) (or (2.4)-(2.5)) are defined by the initial values $\mathbf{S}(X,0,\epsilon)$, $\mathbf{S}_T(X,0,\epsilon)$, $\mathbf{n}(X,0,\epsilon)$ and exist in the time interval $[0,T_0)$ defined by the Whitham system (2.20) and the initial data $\mathbf{S}_{(0)}(X,0) = \mathbf{S}(X,0,0)$, $\mathbf{S}_{(0)T}(X,0) = \mathbf{S}_T(X,0,0)$, and $\mathbf{n}_{(0)}(X,0) = \mathbf{n}(X,0,0)$.

The deformation procedure.

Let us note now that the series (2.10) (or (2.4)-(2.5)) give in fact the one-parametric formal solutions of (1.1) with a parameter ϵ. Let us rewrite now the solutions (2.10) in the form which gives the concrete (formal) solution of (1.1) and is not connected with the additional one-parametric ϵ-family including this given solution. We omit now the ϵ-dependence of functions $\mathbf{S}(X,T,\epsilon)$, $\mathbf{n}(X,T,\epsilon)$ (or put formally $\epsilon = 1$) and say that the Whitham solution is defined now by functions $\mathbf{S}(X,T)$, $\mathbf{n}(X,T)$ determined by the initial values $\mathbf{S}(X,0)$, $\mathbf{S}_T(X,0)$, and $\mathbf{n}(X,0)$. (Let us keep here the notations X and T for the spatial and time coordinates just to emphasize that we consider the "slow" functions $\mathbf{S}_X(X,T)$, $\mathbf{S}_T(X,T)$, $\mathbf{n}(X,T)$.)

Thus, we define now the Whitham solution as the solution of the system

$$F^i(\boldsymbol{\varphi},\, \boldsymbol{\varphi}_T,\, \boldsymbol{\varphi}_X,\, \boldsymbol{\varphi}_{TT},\, \boldsymbol{\varphi}_{XT},\, \boldsymbol{\varphi}_{XX}, \dots) = 0 \quad , \quad i = 1, \dots, n \tag{2.21}$$

having the form

[3] Let us remind that we assume that all systems (2.13) are solvable if the corresponding orthogonality conditions are satisfied.

$$\phi^i(\boldsymbol{\theta}, X, T) \;=\; \Phi^i\left(\mathbf{S}(X,T) + \boldsymbol{\theta},\ \mathbf{S}_X(X,T),\ \mathbf{S}_T(X,T),\ \mathbf{n}(X,T)\right) \;+$$
$$+ \sum_{k\geq 1} \Phi^i_{(k)}\left(\mathbf{S}(X,T) + \boldsymbol{\theta},\ X,\ T\right) \tag{2.22}$$

where the functions $\Phi^i_{(k)}$:

1) Are 2π-periodic with respect to each θ^α;

2) Have degree k (introduced below);

3) Satisfy the normalization conditions (2.11)-(2.12) which will be written now in the form:

$$\int_0^{2\pi} \!\!\dots \int_0^{2\pi} \sum_{i=1}^{n} \Phi^i_{\theta^\alpha}\left(\boldsymbol{\theta},\ \mathbf{S}_X,\, \mathbf{S}_T,\, \mathbf{n}\right)\ \Phi^i_{(k)}(\boldsymbol{\theta}, X, T)\ \frac{d^m\theta}{(2\pi)^m} \;=\; 0 \tag{2.23}$$

$$\int_0^{2\pi} \!\!\dots \int_0^{2\pi} \sum_{i=1}^{n} \Phi^i_{n^l}\left(\boldsymbol{\theta},\ \mathbf{S}_X,\, \mathbf{S}_T,\, \mathbf{n}\right)\ \Phi^i_{(k)}(\boldsymbol{\theta}, X, T)\ \frac{d^m\theta}{(2\pi)^m} \;=\; 0 \tag{2.24}$$

$k \geq 1, \quad (\alpha \,=\, 1, \dots, m, \quad l \,=\, 1, \dots, s).$

Let us introduce now the gradation used for the formal expansion (2.22). Namely, for the systems (1.1) having the non-degenerate hyperbolic Whitham system (2.20) we put the following gradation on the functions $\mathbf{S}_X(X,T)$, $\mathbf{S}_T(X,T)$, $\mathbf{n}(X,T)$ and their derivatives :

1) The functions $k^\alpha(X,T) \,=\, S^\alpha_X(X,T)$, $\omega^\alpha(X,T) \,=\, S^\alpha_T(X,T)$, and $n^l(X,T)$ have degree 0;

2) Every differentiation with respect to X adds 1 to the degree of the function;

3) The degree of the product of two functions having certain degrees is equal to the sum of their degrees.

In other words, for the parameters $\mathbf{U} \,=\, (\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ we have the gradation rule of Dubrovin - Zhang type, i.e.
All the functions $f(\mathbf{U})$ have degree 0;
The derivatives U^ν_{kX} have degree k;
The degree of the product of functions having certain degrees is equal to the sum of their degrees.

We put now the evolution conditions to the functions $\mathbf{S}(X,T)$, $\mathbf{n}(X,T)$ having the form :

$$S^\alpha_{TT} \;=\; \sum_{k\geq 1} \sigma^\alpha_{(k)}\left(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n},\, \mathbf{k}_X,\, \boldsymbol{\omega}_X,\, \mathbf{n}_X,\, \dots\right) \tag{2.25}$$

$$n^l_T \;=\; \sum_{k\geq 1} \eta^l_{(k)} \left(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n},\, \mathbf{k}_X,\, \boldsymbol{\omega}_X,\, \mathbf{n}_X,\, \dots\right) \tag{2.26}$$

where $\sigma^\alpha_{(k)}$, $\eta^l_{(k)}$ are general polynomials in derivatives $\mathbf{k}_X$, $\boldsymbol{\omega}_X$, $\mathbf{n}_X$, $\mathbf{k}_{XX}$, $\boldsymbol{\omega}_{XX}$, $\mathbf{n}_{XX}$, ... (with coefficients depending on $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$) having degree k.

We now substitute the expansion (2.22) in the system (2.21) and use the relations (2.25)-(2.26) to remove all the time derivatives of parameters $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$. After that we can divide the system (2.21) into the terms of certain degrees and try to find recursively all the terms $\mathbf{\Phi}_{(k)}(\boldsymbol{\theta}, X, T)$ for $k = 1, 2, \dots$.

It is easy to see again that for any $\mathbf{\Phi}_{(k)}(\boldsymbol{\theta}, X, T)$ we will have the linear system analogous to (2.7), (2.13), i.e.

$$\hat{L}^i_{[\mathbf{S}(X,T),\mathbf{n}(X,T)]j}\; \Phi^i_{(k)}(\boldsymbol{\theta}, X, T) \;=\; \hat{f}^i_{(k)}(\boldsymbol{\theta}, X, T) \tag{2.27}$$

where $\hat{\mathbf{f}}_{(k)}(\boldsymbol{\theta}, X, T)$ is the discrepancy having degree k according to the definition above.

We have to put again the orthogonality conditions

$$\int_0^{2\pi}\!\!\dots\int_0^{2\pi} \kappa^{(q)}_{[\mathbf{S},\mathbf{n}]i}(\boldsymbol{\theta}, X, T)\; \hat{f}^i_{(k)}(\boldsymbol{\theta}, X, T)\; \frac{d^m\theta}{(2\pi)^m} \;\equiv\; 0 \tag{2.28}$$

on the functions $\hat{f}^i_{(k)}(\boldsymbol{\theta}, X, T)$ and then find the unique $\mathbf{\Phi}_{(k)}(\boldsymbol{\theta}, X, T)$ satisfying the normalization conditions (2.23)-(2.24).

It's not difficult to prove the following Lemma:

Lemma 2.5.
1) For any system (1.1) having the non-degenerate hyperbolic Whitham system (2.20) the functions $\sigma^\alpha_{(k)}$, $\eta^l_{(k)}$ are uniquely determined by the orthogonality conditions (2.28) in the order k.
2) The functions $\sigma^\alpha_{(1)}$, $\eta^l_{(1)}$ give the Whitham system (2.18)-(2.19) for the functions $\mathbf{S}_{(0)}(X, T)$, $\mathbf{n}_{(0)}(X, T)$. [4]

Proof.
Indeed, using Lemma 2.1 it is easy to see that the functions $\hat{f}^i_{(1)}(\boldsymbol{\theta}, X, T)$ coincide with the functions $\tilde{f}^i_{(1)}(\boldsymbol{\theta}, X, T)$ introduced in (2.13) after the replacement of functions $\mathbf{S}_{(0)}(X, T)$, $\mathbf{n}_{(0)}(X, T)$ by $\mathbf{S}(X, T)$, $\mathbf{n}(X, T)$. Comparing then the orthogonality conditions (2.28) with (2.14) we get the second part of the Lemma.

[4] In fact, the functions $\mathbf{S}(X, T, \epsilon)$, $\mathbf{n}(X, T, \epsilon)$ introduced previously satisfy the full system (2.25)-(2.26).

To prove the first part we just note that all $\sigma^{\alpha}_{(k)}$, $\eta^{l}_{(k)}$ arise in the k-th order of system (2.21) "in the same way". We can conclude then that the orthogonality conditions (2.28) in the k-th order always contain the functions $\sigma^{\alpha}_{(k)}$, $\eta^{l}_{(k)}$ in one particular way which coincides with the appearance of $\sigma^{\alpha}_{(1)}$, $\eta^{l}_{(1)}$ in the Whitham system arising for $k = 1$. From the definition of the non-degenerate Whitham system we now obtain the first part of the Lemma.

Lemma 2.5 is proved.

Definition 2.1.
We call the system (2.25)-(2.26) or the equivalent system

$$k^{\alpha}_{T} \;\; = \;\; \omega^{\alpha}_{X}$$

$$\omega^{\alpha}_{T} \;\; = \;\; \sum_{k \geq 1} \sigma^{\alpha}_{(k)} \left(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n},\, \mathbf{k}_{X},\, \boldsymbol{\omega}_{X},\, \mathbf{n}_{X},\, \dots \right) \qquad (2.29)$$

$$n^{l}_{T} \;\; = \;\; \sum_{k \geq 1} \eta^{l}_{(k)} \left(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n},\, \mathbf{k}_{X},\, \boldsymbol{\omega}_{X},\, \mathbf{n}_{X},\, \dots \right)$$

the deformation of the Whitham system (2.18)-(2.19) (or (2.20)).

The functions $k^{\alpha}(X,T)$, $\omega^{\alpha}(X,T)$, $n^{l}(X,T)$ are the "slow" functions of the variables x and t and the system (2.29) gives the analog of the "low-dispersion" expansion in our case. The asymptotic solutions (2.22) of the initial system (2.21) are parameterized by the asymptotic solutions $k^{\alpha}(X,T)$, $\omega^{\alpha}(X,T)$, $n^{l}(X,T)$ of the system (2.29) and arbitrary (constant) initial phases θ^{α}_{0}. As follows from our considerations above the solutions (2.22) give all the "particular solutions" (2.4)-(2.5), however, they do not contain the additional information about the one-parametric ϵ-family given by (2.4)-(2.5).

Remark 1.

Let us note that the full set of parameters of m-phase solutions of (1.1) is given by $\mathbf{k}$, $\boldsymbol{\omega}$, $\mathbf{n}$, and $\boldsymbol{\theta}_{0}$. However the functions $\boldsymbol{\theta}_{0}(X,T)$ do not present as the parameters of solutions (2.22) in this approach. This shows in fact that the introduction of the functions $\theta^{\alpha}_{0}(X,T)$ do not give "new" formal solutions of (1.1) and is responsible for the additional ϵ-dependence of one-parametric families (2.4)-(2.5). Here they are "absorbed" by the total phase $\mathbf{S}(X,T)$ connected with the "particular" formal solution of (1.1).

Remark 2.

In our consideration we fixed some functions $\Phi^{i}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ on the family Λ (for each $(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$) as having zero initial phases. However, the choice of the functions $\Phi^{i}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ is not unique. In particular, the natural change of the functions $\Phi^{i}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ on Λ can be written in the form

$$\mathbf{\Phi}^{\prime}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n}) \;\; = \;\; \mathbf{\Phi}(\boldsymbol{\theta} \, + \, \boldsymbol{\theta}_{0}(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n}),\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n}) \tag{2.30}$$

where $\theta_{0}^{\alpha}(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ are arbitrary smooth functions.

It's not difficult to see also that the system (2.29) depends on the choice of the functions $\Phi^{i}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ in the high ($k \geq 2$) orders.

Let us give here the definition given by B.A. Dubrovin and Y. Zhang ([63, 65]) and connected with the "equivalence" of different infinite (or finite) systems.

Definition 2.2 (B.A. Dubrovin, Y. Zhang).
Consider the system of the form

$$U^{\nu}_{T} \;\; = \;\; \sum_{k \geq 1} V^{\nu}_{(k)} \left(\mathbf{U},\, \mathbf{U}_{X},\, \mathbf{U}_{XX},\, \dots \right) \quad , \qquad \nu = 1, \dots, N \tag{2.31}$$

for arbitrary parameters U^{ν} where all $V^{\nu}_{(k)}$ are smooth functions polynomial in $\mathbf{U}_{X}$, $\mathbf{U}_{XX}$, $\dots$ and having degree k. We say that two systems (2.31) are connected by the triviality transformation (or equivalent) if they are connected by the formal substitution

$$\tilde{U}^{\nu} \;\; = \;\; \sum_{k \geq 0} \tilde{U}^{\nu}_{(k)} \left(\mathbf{U},\, \mathbf{U}_{X},\, \mathbf{U}_{XX},\, \dots \right)$$

where all $\tilde{U}^{\nu}_{(k)}$ are smooth functions polynomial in $\mathbf{U}_{X}$, $\mathbf{U}_{XX}$, $\dots$ and having degree k.

Let us say actually that the definition of B.A. Dubrovin and Y. Zhang is applied usually to the whole integrable hierarchies and plays the important role in the classification of integrable hierarchies having the form (2.31). We will prove here the following Lemma:

Lemma 2.6.
The deformations of the Whitham system (2.29) written for the initial functions $\mathbf{\Phi}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ and $\mathbf{\Phi}^{\prime}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ connected by the transformation (2.30) are equivalent in Dubrovin - Zhang sense.

Proof.
Let us prove first the following statement:
For any transformation (2.30) there exists a unique change of functions $S^{\alpha}(X)$, $n^{l}(X)$:

$$S^{\prime\alpha} \;\; = \;\; S^{\alpha} \; - \; \theta_{0}^{\alpha}(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n}) \; + \; \sum_{k \geq 1} S^{\alpha}_{(k)} \left(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n},\, \mathbf{k}_{X}, \dots \right)$$

$$n^{\prime l} \;\; = \;\; n^{l} \; + \; \sum_{k \geq 1} n^{l}_{(k)} \left(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n},\, \mathbf{k}_{X}, \dots \right) \tag{2.32}$$

such that:

1) All $\mathbf{S}_{(k)}$, $\mathbf{n}_{(k)}$ are polynomial in derivatives of $(\mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ and have degree k;

2) For any solution (2.22) $(\boldsymbol{\phi}_{[\mathbf{S},\mathbf{n}]}(\boldsymbol{\theta}, X, T))$ of (2.21) the functions $\Phi^{\prime i}(\mathbf{S}^{\prime}(X,T) + \boldsymbol{\theta}, \mathbf{S}^{\prime}_{X}, \mathbf{S}^{\prime}_{T}, \mathbf{n}^{\prime})$ satisfy the normalization conditions

$$\int_0^{2\pi} \!\! \dots \int_0^{2\pi} \sum_{i=1}^{n} \Phi^{\prime i}_{\theta^\alpha}(\mathbf{S}^{\prime} + \boldsymbol{\theta}, \mathbf{S}^{\prime}_{X}, \mathbf{S}^{\prime}_{T}, \mathbf{n}^{\prime}) \; \phi^{i}_{[\mathbf{S},\mathbf{n}]}(\boldsymbol{\theta}, X, T) \; \frac{d^m\theta}{(2\pi)^m} \quad \equiv \quad 0 \tag{2.33}$$

$$\int_0^{2\pi} \!\! \dots \int_0^{2\pi} \sum_{i=1}^{n} \Phi^{\prime i}_{n^{\prime l}}(\mathbf{S}^{\prime} + \boldsymbol{\theta}, \mathbf{S}^{\prime}_{X}, \mathbf{S}^{\prime}_{T}, \mathbf{n}^{\prime}) \left[\phi^{i}_{[\mathbf{S},\mathbf{n}]}(\boldsymbol{\theta}, X, T) - \Phi^{\prime i}(\mathbf{S}^{\prime} + \boldsymbol{\theta}, \mathbf{S}^{\prime}_{X}, \mathbf{S}^{\prime}_{T}, \mathbf{n}^{\prime}) \right] \frac{d^m\theta}{(2\pi)^m} \equiv 0 \tag{2.34}$$

For the proof of this statement let us note first that we can always express all the time derivatives of $\mathbf{k}$, $\boldsymbol{\omega}$, and $\mathbf{n}$ using the system (2.29) in terms of X-derivatives of these functions. Using this procedure we try to find the transformation (2.32) recursively in all degrees by substitution of (2.32) in (2.33)-(2.34). It's not difficult to check then that the functions $S^{\alpha}_{(k)}$, $n^{l}_{(k)}$ are defined in the k-th order of (2.33)-(2.34) from a linear system. The matrix of this linear system coincides with the Gram matrix of functions Φ_{θ^α}, Φ_{n^l} at every X and T. Thus, for the full non-degenerate family of m-phase solutions of (1.1) this system has a unique solution at every degree k. The transformation (2.32) is evidently invertible in sense of the infinite series (polynomial w.r.t. derivatives of ($\mathbf{k}$, $\boldsymbol{\omega}$, $\mathbf{n}$)) so we can also express the functions $(\mathbf{S}, \mathbf{n})$ in terms of $(\mathbf{S}^{\prime}, \mathbf{n}^{\prime})$.

We can try to use now the functions $\Phi^{\prime i}(\mathbf{S}^{\prime} + \boldsymbol{\theta}, \mathbf{S}^{\prime}_{X}, \mathbf{S}^{\prime}_{T}, \mathbf{n}^{\prime})$ as the zero approximation in the $(\mathbf{S}^{\prime}, \mathbf{n}^{\prime})$-expansion of the corresponding solution (2.22). It is not difficult to see that the difference

$$\Phi^{\prime i}(\mathbf{S}^{\prime} + \boldsymbol{\theta}, \mathbf{S}^{\prime}_{X}, \mathbf{S}^{\prime}_{T}, \mathbf{n}^{\prime}) \;\; - \;\; \phi^{i}_{[\mathbf{S},\mathbf{n}]}(\boldsymbol{\theta}, X, T)$$

can be represented as the infinite series polynomial w.r.t. derivatives of ($\mathbf{k}$, $\boldsymbol{\omega}$, $\mathbf{n}$) and starting with the terms of degree 1. After the expression of the functions ($\mathbf{k}$, $\boldsymbol{\omega}$, $\mathbf{n}$) in terms of ($\mathbf{k}^{\prime}$, $\boldsymbol{\omega}^{\prime}$, $\mathbf{n}^{\prime}$) in this difference we get finally the $(\mathbf{S}^{\prime}, \mathbf{n}^{\prime})$-expansion of the solution $\phi^{i}_{[\mathbf{S},\mathbf{n}]}(\boldsymbol{\theta}, X, T)$.

The functions $(\mathbf{S}^{\prime}, \mathbf{n}^{\prime})$ satisfy now the new deformed Whitham system (2.25)-(2.26) corresponding to the choice of the functions $\boldsymbol{\Phi}^{\prime}(\boldsymbol{\theta}, \mathbf{k}^{\prime}, \boldsymbol{\omega}^{\prime}, \mathbf{n}^{\prime})$ as the functions of zero approximation.

Easy to see also that the transformation (2.32) remains polynomial in X-derivatives of $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ after the expression of time derivatives of $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ using the system (2.29).

We obtain then that the transformation (2.32) gives a "triviality" connection between the systems (2.29) written for the initial functions $\boldsymbol{\Phi}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$ and $\boldsymbol{\Phi}^{\prime}(\boldsymbol{\theta},\, \mathbf{k},\, \boldsymbol{\omega},\, \mathbf{n})$.

Lemma 2.6 is proved.

At the end let us note again that the Lemma 2.6 is important in fact for the integrable hierarchies rather than for the one particular system (1.1) according to Dubrovin - Zhang approach to classification of integrable systems.

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